Transport of turbulent vorticity increments

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Starting with the Navier-Stokes equations, a transport equation is written for the sum of the squared vorticity increments in homogeneous isotropic turbulence. This equation is compared with that for the sum of the squared velocity increments; whereas the latter equation exhibits a linear dependence on separation, the former does not. In the limit of a negligibly small separation, the new equation expresses a balance between the production and dissipation of the mean square vorticity gradient. All terms in the equation have been measured using a three-component vorticity probe in the self-preserving region of a low Reynolds number turbulent wake. [S1063-651X(98)10904-2]

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I. INTRODUCTION

An important equation in the context of isotropic turbulence is that written by Kolmogorov [1] using the Kármán-Howarth equation [2] as a point of departure. This equation describes the transport of $(\delta u_1)^2$, where $\delta u_1 \equiv u_1(x_1+r_1) - u_1(x_1)$ is the longitudinal velocity increment (x_1 and r_1 represent the longitudinal coordinate and separation, respectively) and has received reasonably good experimental support. Recently, the equation for $(\delta u_1)^2$ was generalized [3] to an equation for $(\delta u_i)^2$, with repeated subscripts implying summation. We inquire here into the form of the equation that describes the transport of $(\delta\omega_i)^2$, where ω_i is the vorticity fluctuation. Information on the vorticity field is desirable for a number of reasons, not the least of which is the connection between the internal dynamics of turbulence and the self-amplifying characteristic of vorticity [4,5]. While the characteristics of the velocity increment have been extensively studied, especially in connection with the effect of small scale intermittency, little is known about the statistics of $(\delta\omega_i)^2$ or just one of its components. The derivation of the equation for $\delta\omega_i^2$ is presented in Sec. II; both the equation and its limiting form (when $r_1 \to 0$) are discussed by comparison to the $(\delta u_i)^2$ equation. Measurements of all three components of the fluctuating ω_i vorticity vector were made in a turbulent wake with a new vorticity probe [6]. Details of the experiment are given in Sec. III. Results for moments of $\langle(\delta\omega_i)^2\rangle$ and for all the terms in the transport equation for $(\delta\omega_i)^2$ are presented in Sec. IV.

II. TRANSPORT EQUATION FOR $(\delta\omega_I)^2$

The transport equation for the instantaneous vorticity fluctuation ω_i at point x_i may be written as [7,8]

$$\partial_t\omega_i + u_\alpha\partial_\alpha\omega_i = \omega_\alpha\partial_\alpha u_i + \nu\partial_\alpha^2\omega_i , \tag{1}$$

where the notation is such that $\partial_t \equiv \partial/\partial t$, $\partial_\alpha \equiv \partial/\partial x_\alpha$, and $\partial_\alpha^2 \equiv \partial^2/\partial x_\alpha^2$. By subtracting Eq. (1) from the corresponding vorticity equation at point $x_i^+ \equiv x_i + r_i$, the difference is

$$\underbrace{\partial_t(\delta\omega_i)}_{1} + \underbrace{\delta u_\alpha\partial_\alpha^+(\delta\omega_i)}_{2} + \underbrace{u_\alpha(\partial_\alpha^+ + \partial_\alpha)\delta\omega_i}_{3} = \underbrace{\omega_\alpha^+\partial_\alpha^+(\delta u_i) + \omega_\alpha\partial_\alpha(\delta u_i)}_{4} + \underbrace{\nu(\partial_\alpha^{+2} + \partial_\alpha^2)\delta\omega_i}_{5} \tag{2}$$

where $\delta\omega_i (\equiv \omega_i^+ - \omega_i)$ is the vorticity increment, $\delta u_i (\equiv u_i^+ - u_i)$ is the velocity increment, and the superscript "+" refers to quantities at x_i^+. We used the fact that coordinates x_i and x_i^+ are independent, i.e., $\partial(\ )^+/\partial x_\alpha \equiv 0$ and $\partial(\ )/\partial x_\alpha^+ \equiv 0$. After multiplying Eq. (2) by $2\delta\omega_i$ and averaging (angular brackets denote ensemble averaging), terms 1–5 in Eq. (2) become

$$2\langle \delta\omega_i\partial_t(\delta\omega_i)\rangle = \partial_t\langle(\delta\omega_i)^2\rangle, \tag{3a}$$

$$2\langle \delta\omega_i\delta u_\alpha\partial_\alpha^+(\delta\omega_i)\rangle = \frac{\partial}{\partial r_\alpha}\langle \delta u_\alpha(\delta\omega_i)^2\rangle, \tag{3b}$$

$$2\langle \delta\omega_i u_\alpha(\partial_\alpha^+ + \partial_\alpha)\delta\omega_i\rangle = \partial_\alpha^+\langle u_\alpha(\delta\omega_i)^2\rangle + \partial_\alpha\langle u_\alpha(\delta\omega_i)^2\rangle, \tag{3c}$$

$$\begin{aligned} 2\langle \delta\omega_i\omega_\alpha^+\partial_\alpha^+(\delta u_i)\rangle + 2\langle \delta\omega_i\omega_\alpha\partial_\alpha(\delta u_i)\rangle &= 4\langle \omega_i\omega_\alpha\partial_\alpha u_i\rangle - 2\frac{\partial}{\partial r_\alpha}\langle \omega_i\omega_\alpha^+ u_i^+\rangle \\ &\quad + 2\frac{\partial}{\partial r_\alpha}\langle \omega_i^+\omega_\alpha u_i\rangle, \end{aligned} \tag{3d}$$

$$2\nu\langle\delta\omega_i\partial_\alpha^{+2}(\delta\omega_i)\rangle+2\nu\langle\delta\omega_i\partial_\alpha^2(\delta\omega_i)\rangle$$
$$=\nu\langle\partial_\alpha^{+2}(\delta\omega_i)^2\rangle+\nu\langle\partial_\alpha^2(\delta\omega_i)^2\rangle-2\nu\langle[\partial_\alpha^+(\delta\omega_i)]^2\rangle$$
$$-2\nu\langle[\partial_\alpha(\delta\omega_i)]^2\rangle$$
$$=2\nu\frac{\partial^2}{\partial r_\alpha^2}\langle(\delta\omega_i)^2\rangle-4\langle\epsilon_\omega\rangle. \tag{3e}$$

Equation (3c) is zero since, due to the definition $r_i\equiv x_i^+-x_i$,

$$\partial_\alpha^+\langle\ \rangle\equiv\frac{\partial}{\partial r_\alpha}\langle\ \rangle \quad\text{and}\quad \partial_\alpha\langle\ \rangle\equiv-\frac{\partial}{\partial r_\alpha}\langle\ \rangle.$$

In Eq. (3e), $\langle\epsilon_\omega\rangle\equiv\nu\langle(\partial_\alpha\omega_i)^2\rangle$ is the destruction rate of the mean square vorticity. For stationary turbulence at a sufficiently large Reynolds number [9]

$$\langle\omega_i\omega_\alpha\partial_\alpha u_i\rangle=\langle\epsilon_\omega\rangle, \tag{4}$$

viz. $\langle\epsilon_\omega\rangle$ balances the generation of $\langle\omega_i^2\rangle$ due to the interaction between the instantaneous turbulent strain rate and vorticity fluctuations. After combining Eqs. (3a)–(3e), the resulting equation is

$$\frac{\partial}{\partial t}\langle(\delta\omega_i)^2\rangle+\frac{\partial}{\partial r_\alpha}\langle\delta u_\alpha(\delta\omega_i)^2\rangle-4\langle\omega_i\omega_\alpha\partial_\alpha u_i\rangle$$
$$-2\nu\frac{\partial^2}{\partial r_\alpha^2}\langle(\delta\omega_i)^2\rangle+2\frac{\partial}{\partial r_\alpha}\langle\omega_i\omega_\alpha^+u_i^+-\omega_i^+\omega_\alpha u_i\rangle$$
$$+4\langle\epsilon_\omega\rangle=0. \tag{5}$$

An order of magnitude argument suggests that the first term of Eq. (5) should be negligible when the Reynolds number is sufficiently large. With this term ignored, and making use of Eq. (4), Eq. (5) can be simplified to

$$\frac{\partial}{\partial r_\alpha}\langle\delta u_\alpha(\delta\omega_i)^2\rangle-2\nu\frac{\partial^2}{\partial r_\alpha^2}\langle(\delta\omega_i)^2\rangle$$
$$=-2\frac{\partial}{\partial r_\alpha}\langle\omega_i\omega_\alpha^+u_i^+-\omega_i^+\omega_\alpha u_i\rangle. \tag{6}$$

Only homogeneity has been used to obtain Eq. (6). If isotropy is introduced, Eq. (6) can be projected onto the x_1 direction, viz.,

$$\left(\frac{2}{r_1}+\frac{\partial}{\partial r_1}\right)\langle\delta u_1(\delta\omega_i)^2\rangle-2\nu\left(\frac{2}{r_1}+\frac{\partial}{\partial r_1}\right)\frac{\partial}{\partial r_1}\langle(\delta\omega_i)^2\rangle$$
$$=-2\left(\frac{2}{r_1}+\frac{\partial}{\partial r_1}\right)\langle\omega_i\omega_1^+u_i^+-\omega_i^+\omega_1u_i\rangle. \tag{7}$$

Using an argument similar to that by Kármán and Howarth [2] (see also [8]) the only solution of the equation

$$\left(\frac{2}{r_1}+\frac{\partial}{\partial r_1}\right)f(r_1)=0,$$

which has no singularity at $r_1=0$, is $f(r_1)=0$. The solution of Eq. (7) is therefore given by

$$\langle\delta u_1(\delta\omega_i)^2\rangle-2\nu\frac{\partial}{\partial r_1}\langle(\delta\omega_i)^2\rangle=-2\langle\omega_i\omega_1^+u_i^+-\omega_i^+\omega_1u_i\rangle. \tag{8}$$

This equation differs in an important way from the transport equation for $\langle(\delta u_i)^2\rangle$ [3],

$$\langle\delta u_1(\delta u_i)^2\rangle-2\nu\frac{\partial}{\partial r_1}\langle(\delta u_i)^2\rangle=-\tfrac{4}{3}\langle\epsilon\rangle r_1, \tag{9}$$

where $\langle\epsilon\rangle$ is the average turbulent energy dissipation rate. Whereas $\langle\delta u_1(\delta u_i)^2\rangle$ increases linearly with r_1 in the inertial range (IR), $\langle\delta u_1(\delta\omega_i)^2\rangle$ is likely to decrease over this region if the two-point correlations on the right of Eq. (8) become negligible. Such a behavior would be qualitatively consistent with the observation by Antonia *et al.* [10] that two-point vorticity correlations should decrease as $r_1^{-4/3}$ in the IR. These authors noted [10] that $\langle(\delta\omega_i)^2\rangle$ is unlikely to exhibit a power-law behavior in the IR: Fan [11] also noted the absence of a power-law behavior in the spectrum of ω_1. The absence of a dependence on r_1, over the IR, for $\langle\delta u_1(\delta\omega_i)^2\rangle$ would contrast with the existence of such a behavior for two-dimensional turbulence [12,13].

The limiting behavior of Eq. (8) when $r_1\to0$ can be inferred from a Taylor series expansion of Eq. (8) about $r_1=0$. Retaining terms up to order r_1^3, the left side of Eq. (8) reduces to

$$\langle\delta u_1(\delta\omega_i)^2\rangle\simeq\langle\partial_1u_1(\partial_1\omega_i)^2\rangle r_1^3 \tag{10a}$$

and

$$-2\nu\frac{\partial}{\partial r_1}\langle(\delta\omega_i)^2\rangle\simeq-4\nu\langle(\partial_1\omega_i)^2\rangle r_1-2\nu\langle(\partial_1^2\omega_i)^2\rangle r_1^3$$
$$-\frac{8\nu}{3}\langle\partial_1\omega_i\partial_1^3\omega_i\rangle r_1^3. \tag{10b}$$

The term on the right of Eq. (8) can be rewritten as follows:

$$-2\langle\omega_i\omega_1^+u_i^+-\omega_i^+\omega_1u_i\rangle=2\langle(\omega_i^+-\omega_i^-)u_i\omega_1\rangle \tag{11}$$

since

$$\langle\omega_iu_i^+\omega_1^+\rangle=\langle\omega_i^-u_i\omega_1\rangle$$

by virtue of homogeneity (the superscript ‘‘$-$’’ refers to location x_1-r_1). Using Taylor series expansions for ω_i^+ and ω_i^- about $r_1=0$, we can show that, to order r_1^3,

$$2\langle(\omega_i^+-\omega_i^-)u_i\omega_1\rangle\simeq4\langle u_i\omega_1\partial_1\omega_i\rangle r_1+\tfrac{2}{3}\langle u_i\omega_1\partial_1^3\omega_i\rangle r_1^3. \tag{12}$$

Using homogeneity and the isotropic form of the fourth-order tensor $\langle u_i\psi_j\partial_l\omega_k\rangle$, it can be shown that (see Appendix)

$$-\langle u_i\omega_1\partial_1\omega_i\rangle=\langle\omega_i\omega_1\partial_1u_i\rangle. \tag{13}$$

Further, Eq. (4) and the isotropic form of $\langle\partial_j\omega_i\partial_l\omega_k\rangle$ allow the right side of Eq. (13) to be written as

$$\langle\omega_i\omega_1\partial_1u_i\rangle=\nu\langle(\partial_1\omega_i)^2\rangle. \tag{14}$$

As a consequence of Eqs. (13) and (14), the r_1 term in Eq. (12) cancels the r_1 term in relation (10b). This leaves balance (in the limit $r_1 \to 0$) between the r_1^3 terms in relations (10a), (10b), and (12), viz.,

$$\langle \partial_1 u_1 (\partial_1 \omega_i)^2 \rangle - 2\nu \langle (\partial_1^2 \omega_i)^2 \rangle - \frac{8\nu}{3} \langle \partial_1 \omega_i \partial_1^3 \omega_i \rangle = \frac{2}{3} \langle u_i \omega_1 \partial_1^3 \omega_i \rangle. \tag{15}$$

The viscous terms can be combined since

$$\langle \partial_1 \omega_i \partial_1^3 \omega_i \rangle = -\langle (\partial_1^2 \omega_i)^2 \rangle,$$

because of homogeneity. Equation (15) becomes

$$\tfrac{2}{3} \nu \langle (\partial_1^2 \omega_i)^2 \rangle = \tfrac{2}{3} \langle u_i \omega_1 \partial_1^3 \omega_i \rangle - \langle \partial_1 u_1 (\partial_1 \omega_i)^2 \rangle. \tag{16}$$

The limiting form (when $r_1 \to 0$) of Eq. (9) is

$$\tfrac{2}{3} \nu \langle (\partial_1^2 u_i)^2 \rangle = -\langle (\partial_1 u_1)(\partial_1 u_i)^2 \rangle. \tag{17}$$

While Eq. (17) expresses a balance between production and dissipation of the mean square vorticity in isotropic turbulence, Eq. (16) can be interpreted as representing the equality between the generation and destruction of the mean square vorticity gradient, also for isotropic turbulence. Equation (17) has been written, e.g., [14], in terms of u_1.

III. EXPERIMENTAL DETAILS

The three components of ω_i were measured simultaneously with a vorticity probe comprising four X wires (i.e., a total of eight hot wires: a sketch of this probe was given in Zhu and Antonia [6]). Two X wires are in the x_1-x_2 plane and are separated by a distance in the x_3 direction of 2.8 mm. The other two are in the x_1-x_3 plane with a separation in the x_2 direction of 2.5 mm. The lateral separation between inclined wires in each X probe was about 1 mm. The included angle for each X wire was about 100°. The 2.5 μm diameter wires were operated at an overheat ratio of 0.5 in constant temperature circuits. Output voltages from the circuits were passed through buck and gain circuits and low-pass filtered at a cutoff frequency of 800 Hz. Sampling was carried out at a frequency of 2000 Hz using a 12-bit analog to digital converter. The record duration was about 120 sec.

Measurements were made on the center line of the wake generated by a circular cylinder (diameter $d = 6.35$ mm) at a distance of $240d$ downstream of the cylinder. With a free stream velocity U_1 of 3.6 m/s, the wake half-width at the measurement location was 26 mm. The Reynolds number R_λ based on the longitudinal Taylor microscale $\lambda = \langle u_1^2 \rangle^{1/2} / \langle u_{1,1}^2 \rangle^{1/2}$ was equal to 40 and the Kolmogorov length scale $\eta = \nu^{3/4} / \langle \epsilon_{\rm iso} \rangle^{1/4}$ was 0.64 mm (note that the characteristic x_2 or x_3 dimension of the probe is about 4η). The full value of $\langle \epsilon \rangle$, which includes 12 components, can be inferred from the velocity derivatives that are measured with the probe. This value, which was 7% smaller than $\langle \epsilon \rangle_{\rm iso} = 15\nu \langle u_{1,1}^2 \rangle$, was used for estimating η.

The flow choice was dictated partly by the need to have as large a value of η as possible (this can be realized in the

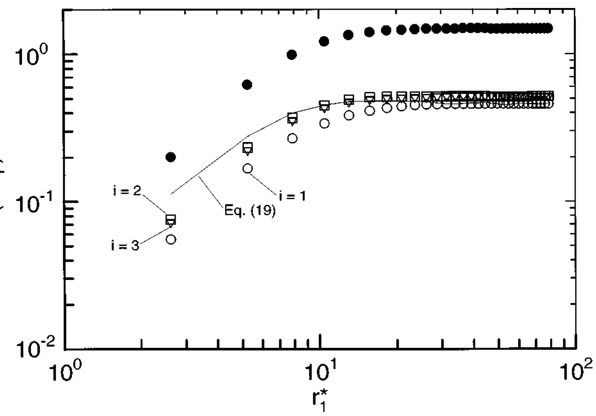

FIG. 1. Second-order moments of vorticity increments in a self-preserving wake. ○: $\langle (\delta\omega_1^*)^2 \rangle$; □: $\langle (\delta\omega_2^*)^2 \rangle$; ▽: $\langle (\delta\omega_3^*)^2 \rangle$; ●: $\langle (\delta\omega_i^*)^2 \rangle$. —: calculation of $\langle (\delta\omega_2^*)^2 \rangle$ or $\langle (\delta\omega_3^*)^2 \rangle$ using Eq. (19).

self-preserving region of a wake at least at small Reynolds numbers) and partly because the small turbulence intensity level in the wake (<3% on the centre line). The latter factor should allow the use of Taylor's hypothesis. This hypothesis was used for converting temporal increments (of u_i or ω_1) into spatial increments or temporal correlations into spatial correlations.

IV. RESULTS

Equation (8) can be nondimensionalized by multiplication with η^2 / U_K^3, viz.,

$$\underbrace{\langle \delta u_1^* (\delta \omega_i^*)^2 \rangle}_{I} - \underbrace{2 \frac{\partial}{\partial r_1^*} \langle (\delta \omega_i^*)^2 \rangle}_{II} = \underbrace{-2 \langle \omega_i^* \omega_1^{*+} u_i^{*+} - \omega_i^{*+} \omega_1^* u_i^* \rangle}_{III} \tag{18}$$

where the asterisk denotes normalization by the Kolmogorov scales η and/or U_K. All terms in Eq. (18) have been measured over the range $3 \lesssim r_1^* \lesssim 50$.

Distributions of $\langle (\delta\omega_i^*)^2 \rangle$ and its components are shown in Fig. 1. Note that $\langle (\delta\omega_2^*)^2 \rangle$ and $\langle (\delta\omega_3^*)^2 \rangle$ are approximately equal and become constant at a smaller value of r_1^* ($\simeq 15$) than $\langle (\delta\omega_1^*)^2 \rangle$. The approximate equality of the variances of the transverse vorticity increments is consistent with local isotropy. It follows from the solenoidality of ω_i that $\langle (\delta\omega_2^*)^2 \rangle$ [or $\langle (\delta\omega_3^*)^2 \rangle$] is related to $\langle (\delta\omega_1^*)^2 \rangle$ in the same way as $\langle (\delta u_2^*)^2 \rangle$ [or $\langle (\delta u_3^*)^2 \rangle$] is related to $\langle (\delta u_1^*)^2 \rangle$, viz.,

$$\langle (\delta\omega_2^*)^2 \rangle = \langle (\delta\omega_3^*)^2 \rangle = \left(1 + \frac{r_1^*}{2} \frac{\partial}{\partial r_1^*} \right) \langle (\delta\omega_1^*)^2 \rangle \tag{19}$$

when local isotropy is assumed. The values of $\langle (\delta\omega_2^*)^2 \rangle$ or $\langle (\delta\omega_3^*)^2 \rangle$ calculated with Eq. (19) exceed the measured values at small r_1^* ($\lesssim 8$). It is unlikely that this disagreement reflects a departure from local isotropy; it is more likely to be due to the imperfect spatial resolution of the probe. The at-

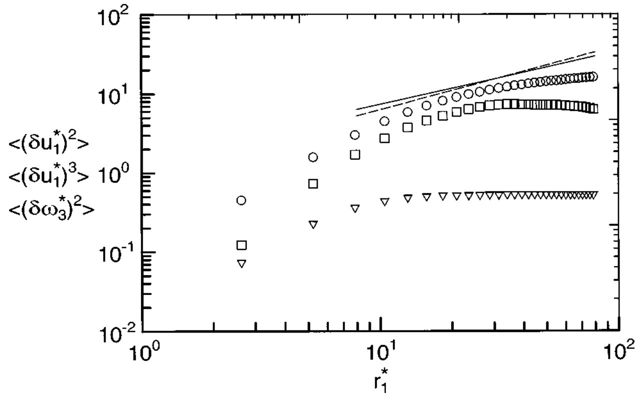

FIG. 2. Comparison of second-order moments of spanwise vorticity increment with second- and third-order moments of the longitudinal velocity increment. ∇: $\langle(\delta\omega_3^*)^2\rangle$; $\bigcirc$: $\langle(\delta u_1^*)^2\rangle$; $\square$: $\langle(\delta u_1^*)^3\rangle$; ——: $\langle(\delta u_1^*)^2\rangle = 2r_1^{*2/3}$; — — —: $\langle(\delta u_1^*)^3\rangle = 4r_1^*/5$.

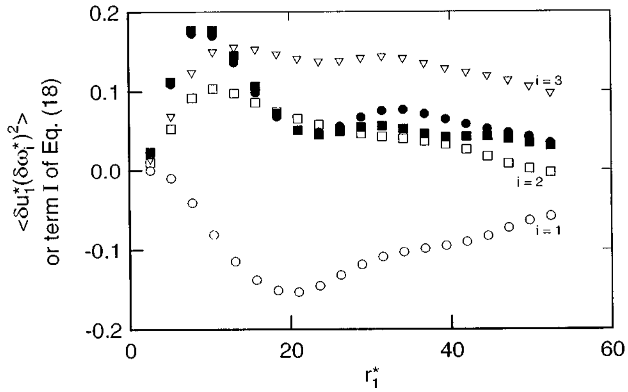

FIG. 3. Mean values of the products of the longitudinal velocity increment and the vorticity increments, i.e., Term I in Eq. (18). $\bigcirc$: $\langle \delta u_1^*(\delta\omega_1^*)^2\rangle$; $\square$: $\langle \delta u_1^*(\delta\omega_2^*)^2\rangle$; ∇: $\langle \delta u_1^*(\delta\omega_3^*)^2\rangle$; $\bullet$: $\langle \delta u_1^*(\delta\omega_i^*)^2\rangle$. The solid squares denote values of $\langle \delta u_1^*(\delta\omega_i^*)^2\rangle$ obtained for a different choice of u_1.

tenuation at small r_1^* (or high wave numbers) is significant and is more important for ω_2 (or ω_3) than ω_1 [15]. No corrections were applied here for this attenuation. The magnitude of $\langle(\delta\omega_i^*)^2\rangle$ attains a constant value of about 1.5 at $r_1^* \simeq 20$. This constancy reflects the small scale nature of vorticity and the relatively rapid decline of the magnitude of the correlation $\langle \omega_i(x_1)\omega_i(x_1+r_1)\rangle$ as r_1 increases. When this correlation becomes negligible,

$$\langle(\delta\omega_i)^2\rangle = 2\langle\omega_i^2\rangle. \tag{20}$$

In homogeneous turbulence,

$$\langle\omega_i^2\rangle = \frac{\langle\epsilon\rangle}{\nu}. \tag{21}$$

Multiplying Eq. (20) by η^2/U_K^2 and using Eq. (21) leads to $\langle(\delta\omega_i^*)^2\rangle = 2$. The measured (constant) value of $\langle(\delta\omega_i^*)^2\rangle$ ($r_1^* \gtrsim 20$) is 25% smaller. This difference reflects the fact that Eq. (21) is only approximately satisfied and possibly also the need to correct for the imperfect spatial resolution of the vorticity probe when $r_1^* \to \infty$ (or infinitely small wave numbers). It is of interest to compare the r_1^* dependence of $\langle(\delta\omega_i)^2\rangle$ with the more established behavior of the velocity structure functions. A comparison is shown in Fig. 2 between $\langle(\delta\omega_3^*)^2\rangle$ and $\langle(\delta u_1^*)^2\rangle$; also included in the figure are the measured values of $\langle(\delta u_1^*)^3\rangle$. Although the present Reynolds number is too small for an inertial range to occur, the increased magnitudes of $\langle(\delta u_1^*)^2\rangle$ and $\langle(\delta u_1^*)^3\rangle$ in the range $10 \lesssim r_1^* \lesssim 30$ seem consistent with the expected $r_1^{*2/3}$ and r_1^* behaviors for these quantities in the range $10 \lesssim r_1^* \lesssim 30$. It is evident, however, that the measured values lie well below the generally accepted value of $2r_1^{*2/3}$ for $\langle(\delta u_1^*)^2\rangle$ and the theoretical (isotropic) value of $4r_1^*/5$ for $\langle(\delta u_1^*)^3\rangle$. The constancy of $\langle(\delta\omega_3^*)^2\rangle$ for $r_1^* \gtrsim 15$ contrasts markedly with the behavior of $\langle(\delta u_1^*)^2\rangle$ and $\langle(\delta u_1^*)^3\rangle$.

The distribution of term I in Eq. (18) is shown in Fig. 3; also shown are the three components of this term. $\langle \delta u_1^*(\delta\omega_1^*)^2\rangle$ is negative and peaks at $r_1^* \simeq 20$; $\langle \delta u_1^*(\delta\omega_2^*)^2\rangle$ and $\langle \delta u_1^*(\delta\omega_3^*)^2\rangle$ are positive with a peak value near $r_1^* \simeq 10$. The sum of the three components (filled in circles) has a main peak at $r_1^* \simeq 10$ and a smaller secondary peak near $r_1^* \simeq 35$. The values of $\langle \delta u_1^*(\delta\omega_i^*)^2\rangle$ in Fig. 3 are based on a particular u_1 signal, obtained from one of the four X wires which make up the vorticity probe; the figure shows that the choice of another u_1 signal (filled in squares) has only a minor effect on the distribution, especially at small r_1^*.

The variation of term III in Eq. (18) and its components is shown in Fig. 4. A possible physical meaning of this term is given by Novikov [4] who identifies it with the self-induced generation of vorticity correlations due to combined convection and stretching effects. The difference $\langle \omega_i^* \omega_1^{*+} u_i^{*+} - \omega_i^{*+} \omega_1^* u_i^*\rangle$ is of opposite sign for $i=1$ than for either $i=2$ or 3 at small values of r_1^*. The latter two distributions follow each other closely. The sum of the three differences (filled in circles) exhibits a positive peak at $r_1^* = 5$ and a smaller negative peak at $r_1^* \simeq 20$. Note that a totally different choice of u_1, u_2, and u_3 signals (solid squares) yielded essentially the same distribution for III (Fig. 4).

The three terms in Eq. (18) are shown in Fig. 5. Term II is of comparable magnitude to term III at small r_1^* although,

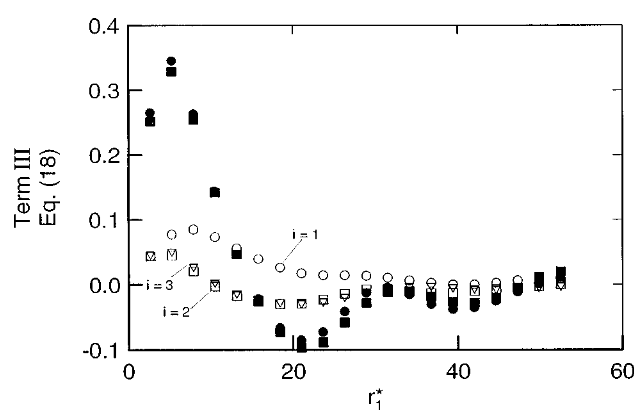

FIG. 4. Velocity-vorticity two-point correlations featured in term III of Eq. (18). $\bigcirc$: $\langle \omega_1^{*+}\omega_1 u_1^* - \omega_1\omega_1^{*+}u_1^{*+}\rangle$; $\square$: $\langle \omega_2^{*+}\omega_1 u_2^* - \omega_2^*\omega_1^{*+}u_2^{*-}\rangle$; ∇: $\langle \omega_3^{*+}\omega_1 u_3^* - \omega_3^*\omega_1^{*+}u_3^{*+}\rangle$; $\bullet$: $2\langle \omega_i^{*+}\omega_1^* u_i^* - \omega_i^*\omega_1^{*+}u_i^{*+}\rangle$. The solid squares represent values of $2\langle \omega_i^{*+}\omega_1^* u_i^* - \omega_i^*\omega_1^{*+}u_i^{*+}\rangle$ obtained for a different choice of (u_1, u_2, u_3).

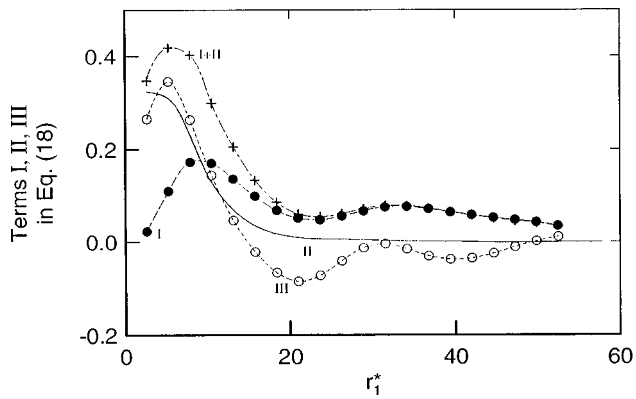

FIG. 5. Terms I, II, and III in Eq. (18). ●: Term I $\equiv\langle \delta u_1^*(\delta\omega_i^*)^2\rangle$; —: term II$\equiv -2\partial/\partial r_1^*\langle(\delta\omega_i^*)^2\rangle$; ○: term III $=2\langle \omega_i^{*+}\omega_1^* u_i^* - \omega_i^*\omega_1^{*+}u_i^{*+}\rangle$; +: sum of terms I and II.

like terms I and III, it should approach zero when $r_1^*\rightarrow 0$. Term III exhibits qualitatively the same behavior as the sum of terms I and II but its magnitude is smaller than that of (I+II) by an almost constant amount (at $r_1^*=5$, where it is nearly maximum, the discrepancy is about 20%). In view of the difficulties associated with the measurements of ω_i, it can be claimed that Fig. 5 provides reasonable support for Eq. (18). Although the high wave number attenuation of ω_i caused by the imperfect spatial resolution of the probe can be corrected [15], the corrections required for terms I, II, and III are much more involved and have not been attempted. It is not unlikely that the combination of these errors may cause a systematic error such as that indicated in Fig. 4.

V. CONCLUSIONS

A transport equation has been derived for the sum of the squared vorticity increments with the assumption of homogeneous and isotropic turbulence. All terms in this equation have been measured in the self-preserving region of a cylinder wake using a three-component vorticity probe. The measurements can be regarded as providing reasonable support for the equation, allowing for the imperfect spatial resolution of the probe. For separations greater than about 20η, the magnitude of all measured terms seems negligible, emphasizing the small-scale nature of vorticity. Accordingly, the peak values of the ‘‘generation’’ term III and the viscous term II in Eq. (18) occur at a separation of about 5η. Speculatively, this may be consistent with the presence of intense vortex filaments with a diameter in the range 4–8η, as has been noted in several simulations, e.g., [16], and a few experiments [17,18].

Although the Reynolds number is too small for an inertial range to be observed in velocity structure functions, the constancy of $\langle(\delta\omega_i^*)^2\rangle$ for $r_1^*\gtrsim 20$ indicated in Fig. 1 is likely to apply at larger values of R_λ. This would be consistent with previous observations that the vorticity spectrum has no power-law behavior at either moderate laboratory values [10] of R_λ or large atmospheric values [11] of R_λ. The constancy of $\langle(\delta\omega_i^*)^2\rangle$ for scales outside the dissipative range certainly contrasts with the expected power-law behavior for the second and higher-order moments of δu_1 (Fig. 2).

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APPENDIX: DERIVATION OF EQS. (13) AND (14)

Assuming homogeneity,

$$\frac{\partial}{\partial x_l}\langle \omega_i\omega_j u_k\rangle = 0$$

and

$$\langle \omega_i\omega_j\partial_l u_k\rangle = -\langle u_k\omega_i\partial_l\omega_j\rangle - \langle u_k\omega_j\partial_l\omega_i\rangle. \tag{A1}$$

After contracting on k and l, Eq. (A1) yields

$$\langle u_k\omega_i\partial_k\omega_j\rangle = -\langle u_k\omega_j\partial_k\omega_i\rangle. \tag{A2}$$

Using the general form of a fourth-order isotropic tensor,

$$\langle u_i\omega_j\partial_l\omega_k\rangle = A\,\delta_{ij}\delta_{kl} + B\,\delta_{il}\delta_{jk} + C\,\delta_{ik}\delta_{jl}\,.$$

Setting $k=l$ and using the solenoidality of ω_k,

$$3A+B+C=0. \tag{A3}$$

Setting $i=l$ and inverting j and k,

$$A+3B+C=0. \tag{A4}$$

It follows from Eqs. (A3) and (A4) that

$$A=B \quad \text{and} \quad C=-4A.$$

Consequently,

$$\langle u_i\omega_j\partial_l\omega_k\rangle - A\langle \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} - 4\,\delta_{ik}\delta_{jl}\rangle \tag{A5}$$

and, using Eq. (A1),

$$\langle \omega_i\omega_j\delta_l u_k\rangle = A(3\,\delta_{ik}\delta_{jl} - 2\,\delta_{kl}\delta_{ij} + 3\,\delta_{jk}\delta_{il}). \tag{A6}$$

For $j=1$ and $l=1$, Eqs. (A5) and (A6) yield

$$-\langle u_i\omega_1\partial_1\omega_i\rangle = \langle \omega_i\omega_1\partial_1 u_i\rangle,$$

i.e., Eq. (13).

Equation (14) follows from Eq. (4), i.e.,

$$\langle \omega_i\omega_\alpha\partial_\alpha u_i\rangle = \nu\langle(\partial_\alpha\omega_i)^2\rangle$$

and the isotropic form of $\langle \partial_j\omega_i\partial_l\omega_k\rangle$, viz.,

$$\langle \partial_j\omega_i\partial_l\omega_k\rangle = D(\,\delta_{ik}\delta_{jl} - \tfrac{1}{4}\,\delta_{ij}\delta_{kl} - \tfrac{1}{4}\,\delta_{il}\delta_{jk}).$$

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